II Geometric Langlands Theory via Derived Algebraic Geometry

5 Introduction to DAG

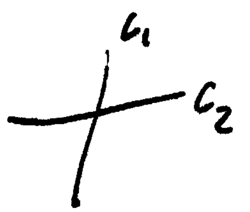
everything $/k$ $\mathrm{char}(k) = 0$

Thm/ Bezout's Theorem $/k = \bar{k}$

Consider $\mathbb{P}^2_k$ projective plane and $C_1, C_2 \subset \mathbb{P}^2_k$ smooth curves of deg m, n intersecting at a finite number of points, Then $mn = \sum_{x \in C_1 \cap C_2} \dim_k (\mathcal{O}_{C_1} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{C_2})_x$ holds

Ex Consider $\mathbb{A}^2_k$

curves $C_1 = \{x\}$ $C_2 = \{y\}$ of deg 1



$$\mathcal{O}_{C_1} \otimes_{\mathbb{A}^2} \mathcal{O}_{C_2}$$
$$= k[x,y]/(x) \otimes k[x,y]/(y)$$
$$\cong k[x,y]/(x,y) \cong k$$

Ex $\mathbb{A}^2$

$C_1 = (y - x^2)$ $C_2 = (y = 0)$

$$k[x,y]/(y-x^2) \underset{k[x,y]}{\otimes} k[x,y]/(y)$$
$$\cong k[x]/(x^2) \quad \dim = 2$$

$\mathcal{O}_{C_1} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{C_2} := \mathcal{O}_{C_1 \cap C_2}$

Question: what happens when $C_1 = C_2$
most degenerate case

Ex $C_1 = (x) \subset \mathbb{P}^2$
$C_2 = (x) \subset \mathbb{P}^2$

$k[x,y]/(x) \otimes_{k[x,y]} k[x,y]/(x)$

one needs a resolution of this as $k[x,y]$ algebra

$$\overset{-1}{\varepsilon k[x,y]} \xrightarrow{d} \overset{0}{k[x,y]} \to k[x,y]/(x)$$

$\deg \varepsilon = -1$ $\quad \| \quad k[x,y,\varepsilon]$ $\qquad f \cdot g = (-1)^{|f||g|} g \cdot f$

$\varepsilon \cdot \varepsilon = (-1)^{1 \cdot 1} \varepsilon \cdot \varepsilon \Rightarrow \varepsilon^2 = 0$

$d(\varepsilon f) = d\varepsilon \cdot f + \underset{0}{\varepsilon \cancel{df}}$

Commutative differential graded algebra,
CDGA $\qquad C^\bullet = \bigoplus_d C^n[-n]$

$(\varepsilon k[x,y] \xrightarrow{x\frac{d}{d\varepsilon}} k[x,y]) \otimes_{k[x,y]} k[x,y]/(x)$ $\qquad C^\bullet$ cochain complex

$= \varepsilon k[x,y]/(x) \xrightarrow{x \cdot d/d\varepsilon = 0} k[x,y]/(x)$ $\qquad C^\bullet[n]$

$= \varepsilon k[y] \xrightarrow{d=0} k[y] = k[y][1] \oplus k[y]$ $\qquad = C^{\bullet+n}$

$\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathcal{O}(\mathbb{P}^2)} \mathcal{O}_{\mathbb{P}^1}$

$$0 \to \underset{\substack{\mathbb{P}2 \\ \mathcal{O}_{\mathbb{P}^2}(-\{x=0\})}}{\mathcal{O}(-1)} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathcal{O}_{\mathbb{P}^1}} \to 0$$

$\mathcal{O}_{\mathbb{P}}(-1)[1] \oplus \mathcal{O}_{\mathbb{P}^1}$

$\chi(\mathcal{O}_{\mathbb{P}^1}(-1)[1] \oplus \mathcal{O}_{\mathbb{P}^1}) = 1$

<u>Note</u>

Grothendieck distinguished

$f=0$ and $f^2=0$.

DAG distinguishes

$f=0$ and $(f=0)^2$

We are led to $CDGA^{\leq 0}$ ~~instead~~ instead of Ring

$\boxed{Sch}$

<u>Defn</u> A derived scheme is a top'l space X with sheaf $\mathcal{O}_X$ valued in $CDGA^{\leq 0}$

s.t. (1) $t_0 = (X, H^0(\mathcal{O}_X))$ is a scheme

(2) $H^i(\mathcal{O}_X)$ is a quasicoherent sheaf over $t_0(X)$

$\forall i \in \mathbb{Z}$

(0 in degree positive)

Ex ① A sheme $(X, \mathcal{O}_X)$ is a derived scheme

② $A \in CDGA^{\leq 0}$ defines a derived scheme

$(\mathrm{Spec}\, H^0 A, A)$

affine derived scheme

$cl\, sch^{aff} \leftrightarrow Ring$

$sch^{aff} \longleftrightarrow CDGA^{\leq 0}$

Rmk

derived scheme: classical scheme

= classical scheme: reduced scheme

Sch is an ∞-category!

For a usual category $\mathcal{C}$, for $X, Y \in \mathcal{C}$

$Hom_{\mathcal{C}}(X, Y)$ is a set

$\uparrow \pi_0$

$Map_{\mathcal{C}}(X, Y)$ is a space

X cl. scheme

$\overset{\sim}{\underset{\text{Yoneda}}{\rightarrow}}\; h_X : (cl\, Sch^{aff})^{op} \to Set$

$S \longmapsto Hom(S, X)$

$\tilde{h}_X : (Sch^{aff})^{op} \to Spc$

$S \longmapsto Map(S, X)$

$X_1 = \operatorname{Spec} A_1 \qquad X_2 = \operatorname{Spec} A_2$

$Y = \operatorname{Spec} B$

$X_1 \times_Y X_2$ fiber product

$\Leftrightarrow \quad A_1 \otimes_B^L A_2$

Expect $h_{X_1 \times_Y X_2}(S) \simeq h_{X_1}(S) \times_{h_Y(S)} h_{X_2}(S)$

Homotopy equivalence

not true!

true w/ $\tilde{h}_X$!

Everything is derived

$\mathrm{Vect}_k :=$ cochain cpx

$\mathrm{comAlg} := \mathrm{CDGA} = \mathrm{comAlg}(\mathrm{Vect})$

QCoh DG category

QCoh abelian category

D DG category

D_X-mod abelian category

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derived scheme is a functor

$(\mathrm{Sch}^{\mathrm{aff}})^{op} \to \mathrm{Spc}$

consider all such functors!

Prestacks! PreStk

These are the most general class of spaces that appear in alg geo.

(so far?)

<u>Ex</u>

- (Betti stack)

  M top'l space, $M \in Spc$

  $M_B : (Sch^{aff})^{op} \to Spc$

  $S \to M$ "constant function"

- (De-Rham stack)

  $\mathcal{Y}$ prestack $\mathcal{Y}_{IR} : (Sch^{aff})^{op} \to Spc$

  $S \mapsto \mathcal{Y}(S^{red})$

- (classifying stack)

  $BG : (Sch^{aff})^{op} \to Spc$

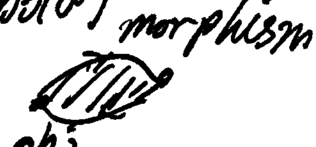
  $S \mapsto G$-bundles on S

  (which is groupoid)

  $Spc \updownarrow$

  $\shortparallel$

  ∞-grd



- (Mapping stack)

  $\mathcal{X}, \mathcal{Y}$ prestacks

  $\underline{Map}(\mathcal{X}, \mathcal{Y})(S) = \underline{Map}(S \times \mathcal{X}, \mathcal{Y})$

can show $\underline{Map}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) = \underline{Map}(\mathcal{X}, \underline{Map}(\mathcal{Y}, \mathcal{Z}))$

Main Example

- X classical scheme

$$\mathrm{Map}(X, B(G)) =: \mathrm{Bun}_G(X)$$

$$\mathrm{Bun}_G(X)(S) = \mathrm{Map}(S \times \cancel{X}, B(G))$$

G-bundles on $S \times X$

- $\mathrm{Map}(X_{dR}, BG) =: \mathrm{Flat}_G(X) =$ de-Rham moduli space of flat G-bundles on X

- M top'l space

$$\underline{\mathrm{Map}}(M_B, BG) =: \mathrm{Loc}_G(M)$$

$=$ character stack

$=$ Betti moduli

2) Quasi-coherent Sheaves

We want DG-category of quasi-coh sheaves on a pre-stack

<u>Defn</u> A DG category is a category enriched over Vect_k = cochain complexes $C_1, C_2 \in \mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ is a complex.

<u>Ex</u>

- Vect ~~dg cat~~

$\underline{\mathrm{Hom}}^{\bullet}_{\mathrm{Vect}}(C^{\bullet}_{d_C}, D^{\bullet}_{d_D})$ is a cochain complex

$$\begin{cases} \mathrm{Hom}^k_{\mathrm{Vect}}(C^{\bullet}, D^{\bullet}) = \prod \mathrm{Hom}(C^i, D^{i+k}) \\ D((\varphi^k)_{k \in \mathbb{Z}}) = (d_D \circ \varphi^k - (-1)^k \varphi^k d_C)_{k \in \mathbb{Z}} \end{cases}$$

$$\begin{tikzcd} C^0 \arrow[r, "d_C"] \arrow[d, "\varphi^0"] & C^1 \arrow[r, "d_C"] \arrow[d, "\varphi^1"] & C^2 \arrow[d, "\varphi^2"] \\ D \arrow[r, "d_D"] & D^1 \arrow[r, "d_D"] & D^2 \end{tikzcd}$$

- DG alg A
  is a DG cat w/ single obj
  $End(\cdot) = A$

- DG alg A
  A-mod . DG cat of DG modules
  $M = \oplus M^n$
  $A^i \cdot M^j \subset M^{i+j}$ $\quad d_M(a\cdot m) = d_A a \cdot m + (-1)^{|a|} a \cdot d_M m$

default assumption on DG categories

$\boxed{DGCat_{cont}}$
- co-complete: has all colimits
- pre-triangulated: $Ho(C)$ is triangulated

  obj. are the same
  morphisms $= H^0(\cdots)$

- functors are continuous
  btwn DG-cat $\quad$ = preserves colimits

$A \xrightarrow{f} B$
$\mathrm{spec}\, B \to \mathrm{spec}\, A$
$f_*: B\text{-mod} \to A\text{-mod}$
$f^*: A\text{-mod} \to B\text{-mod}$ $\otimes_A$ $\otimes_B$
$M \mapsto f^*M = B \otimes_A M$

A-mod-mod $\quad$ functor

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{id \otimes a.} & A \otimes M \\ \downarrow{\scriptstyle a \otimes id.} & & \downarrow{\scriptstyle a.} \\ A \otimes M & \xrightarrow{a.} & M \end{array}$$

Exer $\quad$ $A\text{-mod} \otimes A\text{-mod} \otimes B\text{-mod} \longrightarrow$

$\downarrow \qquad\qquad \downarrow$

$\longrightarrow$

$\Leftrightarrow$ projection formula

$\mathcal{Y}$ -prestack $\qquad$ $S = \operatorname{Spec} A$

$QC(S) = A\text{-mod}$

$$QC(\mathcal{Y}) := \lim_{\left(\substack{S \xrightarrow{y} \mathcal{Y} \\ S \in Sch^{aff}}\right)^{op}} QC(S) \quad \text{in } DG\,Cat_{cont}.$$

$\Downarrow$

$\mathcal{F} \qquad (S, y) \rightsquigarrow \mathcal{F}_{S,y} \in QC(S)$

$S' \xrightarrow{f} S \xrightarrow{y} \mathcal{Y} \rightsquigarrow \mathcal{F}_{S',y} \simeq f^* \mathcal{F}_{S,y}$

A

Rmk 1. This defn. is different from the usual defn. even for a classical scheme! When they are comparable, they coincide.

Formal Properties 1

- $\mathcal{Y} = \operatorname{Spec} A \rightsquigarrow QC(\mathcal{Y}) = A\text{-mod}$
- $\mathcal{O}_{\mathcal{Y}} \Leftrightarrow \{ \mathcal{O}_S \in QC(S) \}$

$f^* : A\text{-mod} \longrightarrow B\text{-mod}$

$M \mapsto B \otimes_A M$

$A \mapsto B$

$QC(\mathcal{Y})$ is a symm monoidal category w/ $\mathcal{O}_{\mathcal{Y}}$ as unit.

$\mathcal{X} \xrightarrow{f} \mathcal{Y}$ map of prestacks

$f^*: QC(\mathcal{Y}) \to QC(\mathcal{X})$
$\mathcal{F} \to f^*\mathcal{F}$

$$\begin{tikzcd} S \arrow[r, "x"] \arrow[dr, "f\circ x"'] & \mathcal{X} \arrow[d, "f"] \\ & \mathcal{Y} \end{tikzcd} \qquad \mathcal{F}_{S, f\circ x} = (f^*\mathcal{F})_{S,x}$$

$QC^*: \text{PreStk} \to \text{DGCat}_{\text{cont.}}$
$\mathcal{Y} \to QC(\mathcal{Y})$
$\mathcal{X} \xrightarrow{f} \mathcal{Y} \to f^*: QC(\mathcal{Y}) \to QC(\mathcal{X})$

How about f_*

<u>Thm</u> (Adjoint Functor thm.)
① Any cont. Functor admits right adj. (cont.)
② Any functor preserving limits admits left adjoint.

$$\begin{aligned} \text{Hom}(F(\text{colim}\, X_i), Y) &= \text{Hom}(\text{colim}\, X_i, GY) \\ &= \lim \text{Hom}(X_i, GY) \\ &= \lim \text{Hom}(F(X_i), Y) \\ &= \text{Hom}(\text{colim}\, F(X_i), Y) \end{aligned}$$

f^* cont. $\rightsquigarrow f_*: QC(\mathcal{X}) \to QC(\mathcal{Y})$
not cont. in general

For

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "g'"] \arrow[d, "f'"'] & \mathcal{Y}_1 \arrow[d, "f"] \\ \mathcal{X}_2 \arrow[r, "g"'] & \mathcal{Y}_2 \end{tikzcd}$$

$QC(\mathrm{Flat}_G)$

$\exists\ g^* \circ f_* \to f'_* \circ g^*$ base change morphism

adj $\quad id \to g'_* \circ g'^*$

$$f_* \to f_* g'_* \circ g'^*$$
$$\searrow \qquad \Vert$$
$$g_* f'_* g'^*$$

use adj $\to$ $\boxed{g^* f_* \to f'_* g'^*}$

How about D-modules?

$\mathcal{Y}$ prestack

$\mathcal{Y}_{dR}$ de-Rham stack (recall D-mod is QC + Flat connection)

$$QC(\mathcal{Y}_{dR}) =: D(\mathcal{Y})$$

Rmk. $\mathcal{Y} = X$ classical smooth scheme

$D(X)$ is equal to DG cat of (left) D_X-modules

$D^{!,!}$ PreStk $\to$ DGCat$_{cont.}$

$\parallel$

$QC^* \circ (-)_{dR}$

$f_* \rightsquigarrow f_{*,dR} : D(\mathcal{X}) \to D(\mathcal{Y})$
de-Rham pushforward

$\underline{\mathcal{X} \to \mathcal{Y}}$

$\mathcal{X} \to \mathcal{X}_{dR}$

$p^* : D(\mathcal{X}) \xrightarrow{obl.} QC(\mathcal{X})$ "oblivion functor"

$\mathcal{X} = X$ classical
this comes from $\mathcal{O}_X \to D_X$

$$\begin{tikzcd} D(\mathcal{Y}) \arrow[r, "obl"] \arrow[d, "f^!"'] \arrow[dr, phantom, "\circlearrowright"] & QC(\mathcal{Y}) \arrow[d, "f^*"] \\ D(\mathcal{X}) \arrow[r, "obl"'] & QC(\mathcal{X}) \end{tikzcd}$$

but $f_{*,dR}$ doesn't give f_* in QC.

$Spec(A)\,(k[\varepsilon]/(\varepsilon^2)) = T[Spec(A)]$

$Spec(A)_{dR}(k[\varepsilon]/(\varepsilon^2)) = Spec(A)(k)$